

# HOMOGENEOUS COORDINATES FOR QUIVER GRASSMANNIANS

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**Introduction.** Let  $M$  be a finite dimensional complex representation of a quiver  $Q$  and  $\underline{e}$  a dimension vector of  $Q$ . As a set, the quiver Grassmannian  $\mathrm{Gr}_{\underline{e}}(M)$  can be defined as the collection of all subrepresentations  $N$  of  $M$  with dimension vector  $\underline{\dim} N = \underline{e}$ . The choice of a totally ordered basis  $\mathcal{B}$  defines an inclusion  $\mathrm{Gr}_{\underline{e}}(M) \rightarrow \mathrm{Gr}(e, d)$  into the ordinary Grassmannian of  $e$ -dimensional subspaces of  $\mathbb{C}^d$  where  $e$  is the total dimension of  $\underline{e}$  and  $d$  is the total dimension of  $M$ . This endows  $\mathrm{Gr}_{\underline{e}}(M)$  with the structure of a complex variety. Composing with the Plücker embedding of  $\mathrm{Gr}(e, d)$  yields an embedding

$$\iota : \mathrm{Gr}_{\underline{e}}(M) \longrightarrow \mathrm{Gr}(e, d) \longrightarrow \mathbb{P}^N$$

into the complex projective space of dimension  $N = \binom{d}{e} - 1$ . The following question suggests itself.

**Problem.** *Can one exhibit equations that describe the image of  $\mathrm{Gr}_{\underline{e}}(M)$  in  $\mathbb{P}^N$ ?*

For the following reasons one might not hope for a uniform answer as in the case of usual Grassmannians. First of all, quiver Grassmannians fail to be homogeneous spaces in general, which means that the shape of the defining equations depend on the choice of the ordered basis  $\mathcal{B}$ . Secondly,  $\mathrm{Gr}_{\underline{e}}(M)$  carries a schematic structure coming from its description as a fibre of the universal quiver Grassmannian of  $Q$  ([1]), and this structure as a scheme is not visible from the viewpoint of a pointwise embedding into  $\mathbb{P}^N$ . Finally, we note that the situation is as bad as possible: every projective scheme is isomorphic to a quiver Grassmannian; cf. [7] or [3].

However, as we will see below, a certain additional geometric information yields a complete list of equations that describes the image of  $\mathrm{Gr}_{\underline{e}}(M)$  in  $\mathbb{P}^N$ .

**Schubert decompositions.** Our approach to the problem is to employ equations that are derived from a Schubert decomposition of  $\mathrm{Gr}_{\underline{e}}(M)$ , as introduced in [5]. We use the following conventions.

To begin with, we assume from now on that the totally ordered basis  $\mathcal{B}$  of  $M$  restricts to a basis  $\mathcal{B}_p = \mathcal{B} \cap M_p$  of  $M_p$  for every vertex  $p$  of the quiver  $Q$ . We write the  $e$ -subsets of  $\mathcal{B}$  as ordered tuples  $I = (i_1, \dots, i_e)$  such that  $i_1 < \dots < i_e$ , and we partially order the  $e$ -subsets of  $\mathcal{B}$  by the rule  $(i_1, \dots, i_e) \leq (j_1, \dots, j_e)$  if and only if  $i_k \leq j_k$  for all  $k = 1, \dots, e$ . The Schubert cell  $C(I)$  of  $\mathrm{Gr}(e, d)$  is the subvariety of all points whose Plücker coordinates  $[\Delta_J]$  in  $\mathbb{P}^N$  satisfy  $\Delta_J = 0$  if  $J \not\leq I$  and  $\Delta_I \neq 0$ . Note that  $I \leq J$  if and only if  $C(I)$  is contained in the closure of  $C(J)$ .

We define the Schubert cell  $C_I^M$  of  $\mathrm{Gr}_{\underline{e}}(M)$  as the intersection of  $\mathrm{Gr}_{\underline{e}}(M)$  with the usual Schubert cell  $C(I)$  inside  $\mathrm{Gr}(e, d)$ . This yields the Schubert decomposition

$$\mathrm{Gr}_{\underline{e}}(M) = \coprod_{I \in \mathcal{B}} C_I^M$$

where  $I$  ranges through all  $e$ -subsets of  $\mathcal{B}$ . Note that the Schubert cells  $C_I^M$  are in general not affine spaces. In particular, many Schubert cells turn out to be empty. For instance, this is the case for all  $I$  that are not of type  $\underline{e}$ , i.e.  $\#(I \cap \mathcal{B}_p) = e_p$  fails for some vertex  $p$  of  $Q$ .

**Equations for the Plücker embedding.** Under a mild assumption, the quiver Grassmannian can be described by the usual Plücker relations, vanishing conditions for irrelevant Plücker coordinates and certain additional equations that we will explain before stating the main result.

The representation  $M$  associates with every arrow  $v : p \rightarrow q$  of  $Q$  a linear map  $M_v : M_p \rightarrow M_q$ , which can be expressed as a matrix  $(\mu_{v,s,t})_{s \in \mathcal{B}_p, t \in \mathcal{B}_q}$  with respect to the basis  $\mathcal{B}$ . The coefficient quiver  $\Gamma = \Gamma(M, \mathcal{B})$  of  $M$  with respect to  $\mathcal{B}$  has vertex set  $\mathcal{B}$  and its arrows  $v : s \rightarrow t$  correspond to the indices of the nonzero matrix coefficients  $\mu_{v,s,t}$  of the linear maps  $M_v$  where  $v$  varies through all arrows of  $Q$ .

A relevant triple is a triple  $(v, s, t)$  where  $v : p \rightarrow q$  is an arrow of  $Q$  and  $s \in \mathcal{B}_p$  and  $t \in \mathcal{B}_q$ . For every relevant triple  $(v, s, t)$  and every  $e$ -subset  $I$ , we define the homogeneous equation

$$F_I(v, s, t) = \sum_{v:s' \rightarrow t'} (-1)^{\epsilon(s', t')} \mu_{v, s', t'} \Delta_s^{s'} \Delta_{t'}^t - \sum_{v:s' \rightarrow t} (-1)^{\epsilon(s', t)} \mu_{v, s', t} \Delta_s^{s'} \Delta_I$$

where the arrows under the sums range through all arrows  $v' : s' \rightarrow t'$  of  $\Gamma$  such that  $v' = v$  and, in the right hand sum,  $t' = t$ . The term  $\Delta_i^j$  stands for  $\Delta_J$  if  $J = I - \{i\} \cup \{j\}$  with  $i \in I$  and  $j \in (\mathcal{B} - I) \cup \{i\}$ , and otherwise  $\Delta_i^j = 0$ . The term  $\epsilon(s', t')$  equals the sum of  $\#\{s'' \in I \mid s' < s'' < s\}$  and  $\#\{t'' \in I \mid t < t'' < t'\}$ .

**Theorem.** *Let  $I$  be an  $e$ -subset of  $\mathcal{B}$  such that  $\text{Gr}_{\underline{e}}(M)$  equals the projective closure of  $C_I^M$ . Then  $\iota$  yields an isomorphism of  $\text{Gr}_{\underline{e}}(M)$  with the closed subscheme of  $\mathbb{P}^N$  that is described by the following list of equations.*

(1) (The Plücker relations)

$$\sum_{k=1}^{e+1} (-1)^k \Delta_{J - \{j_k\}} \Delta_{J' \cup \{j_k\}} = 0$$

for every  $(e+1)$ -subset  $J = (j_1, \dots, j_{e+1})$  and every  $(e-1)$ -subset  $J'$ .

(2) (Trivial Plücker coordinates)

$$\Delta_j^i = 0$$

if  $j < i$  or if  $i \in \mathcal{B}_p$  and  $j \in \mathcal{B}_q$  with  $p \neq q$ .

(3) (The quiver relations)

$$F_I(v, s, t) = \sum_{v:s' \rightarrow t'} (-1)^{\epsilon(s', t')} \mu_{v, s', t'} \Delta_s^{s'} \Delta_{t'}^t - \sum_{v:s' \rightarrow t} (-1)^{\epsilon(s', t)} \mu_{v, s', t} \Delta_s^{s'} \Delta_I = 0$$

for every relevant triple  $(v, s, t)$ .

**Remark on the hypothesis.** If one is only interested in the structure of  $\text{Gr}_{\underline{e}}(M)$  as a complex variety, then it suffices to assume that  $C_I^M$  is dense in  $\text{Gr}_{\underline{e}}(M)$ . For a schematic description of  $\text{Gr}_{\underline{e}}(M)$ , one has to avoid embedded components in the complement of  $C_I^M$ . This latter condition is superfluous if one knows that  $\text{Gr}_{\underline{e}}(M)$  is a reduced scheme. This is, for example, the case if  $M$  is an exceptional representation (see [1, Cor. 4]).

**Proof.** Since the quiver Grassmannian is the projective closure of  $C_I^M$ , the homogeneous relations for  $\text{Gr}_e(M)$  can be deduced by homogenizing equations for the Schubert cell  $C_I^M$ . We exhibit such equations for  $C_I^M$  in the following.

The Schubert cell  $C_I^M$  is contained in the usual Grassmannian  $\text{Gr}(e, d)$ . Therefore the Plücker coordinates satisfy the usual Plücker relations of  $\text{Gr}(e, d)$  as spelled out in part (1) of the theorem.

As a subvariety of the usual Schubert cell  $C(I)$  of  $\text{Gr}_e(M)$ , we can examine  $C_I^M$  in terms of affine coordinates for  $C(I)$ . The points of  $C(I)$  can be identified with complex  $d \times e$ -matrices  $(w_{i,j})_{i \in \mathcal{B}, j \in I}$  in normal form, i.e.  $w_{j,j} = 1$  for all  $j \in I$  and  $w_{i,j} = 0$  if  $i > j$  or if  $i \in I$  and  $i \neq j$ .

The Plücker coordinates can be derived from this matrix representation as  $e \times e$ -minors, i.e.  $\Delta_J = \det(w_{i,j})_{i \in J, j \in I}$ . In particular, we obtain  $\Delta_I = 1$  and  $\Delta_j^i = (-1)^{\eta(i,j)} w_{i,j}$  with  $\eta(i,j) = \#\{k \in I \mid i < k < j\}$ , which allows us to rewrite the following equations in terms of the Plücker coordinates and to homogenize the equations with appropriate powers of  $\Delta_I$ .

The condition  $w_{i,j} = 0$  for  $i > j$  corresponds to the homogeneous equations  $\Delta_i^j = 0$ , which occur in part (2) of the theorem. The conditions  $w_{j,j} = 1$  for  $j \in I$  and  $w_{i,j} = 0$  for  $i \in I$  and  $i \neq j$  do not effect the Plücker coordinates since this normalization merely fixes a basis for the  $e$ -subspace of  $\mathbb{C}^d$  that is represented by the point  $[\Delta_I]$ .

The following is a complete list of equations for  $C_I^M$  as a subscheme of  $C(I)$ , cf. section 1.3 of [6]: for every  $i \in \mathcal{B}_p$  and  $j \in \mathcal{B}_q$  with  $p \neq q$ , we have  $w_{i,j} = 0$ ; and for every relevant triple  $(v, s, t)$ ,

$$E(v, s, t) = \sum_{v:s' \rightarrow t'} \mu_{v,s',t'} w_{s',s} w_{t,t'} - \sum_{v:s' \rightarrow t} \mu_{v,s',t} w_{s',s} = 0$$

where the arrows under the sums vary over all arrows  $(v' : s' \rightarrow t')$  of  $\Gamma$  with  $v' = v$  and, in the right hand sum,  $t' = t$ .

The equations  $w_{i,j} = 0$  correspond to the homogeneous equations  $\Delta_i^j = 0$  as listed in part (2) of the theorem. Using  $\Delta_I = 1$  and  $\epsilon(s', t') = \eta(s', s) + \eta(t, t')$ , the equations  $E(v, s, t) = 0$  correspond to the homogeneous equations

$$F_I(v, s, t) = \sum_{v:s' \rightarrow t'} (-1)^{\epsilon(s', t')} \mu_{v,s',t'} \Delta_s^{s'} \Delta_{t'}^t - \sum_{v:s' \rightarrow t} (-1)^{\epsilon(s', t)} \mu_{v,s',t} \Delta_s^{s'} \Delta_I = 0$$

as they occur in part (3) of the theorem. This finishes the proof.  $\square$

**How to find the dense Schubert cell?** Principally, one can determine the non-empty open Schubert cells algorithmically, using Gröbner bases. If there is a unique open Schubert cell  $C_I^M$  and we intend to ignore embedded components, then  $C_I^M$  is dense in  $\text{Gr}_e(M)$  and we can apply the theorem with respect to  $I$ . Otherwise, one can still derive equations for  $\text{Gr}_e(M)$  by applying the relations of the theorem to each  $I$ , which describes the closures of the Schubert cells  $C_I^M$ , and putting these pieces together.

Note that for a generic choice of an ordered basis  $\mathcal{B}$ , the hypothesis is satisfied for the subset  $I_{\max} \subset \mathcal{B}$  of type  $\underline{e}$  that is maximal among all type  $\underline{e}$ -subsets of  $\mathcal{B}$ . This might, however, not help in practice since one is often interested in particular bases; for instance, bases for which the coefficient quiver  $\Gamma$  has only few arrows. This leads to special arrangements for the embedding  $\text{Gr}_e(M) \rightarrow \mathbb{P}^N$ , and it might happen that  $C_{I_{\max}}^M$  is not dense or even empty. To give an example of an empty maximal cell  $C_{I_{\max}}^M$ , consider the type  $A_2$ -representation  $M = \left[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \right]$  together with its canonical ordered basis and  $\underline{e} = (1, 1)$ .

On the other hand, the choice of a particular basis often provides additional information that is useful for finding a dense Schubert cell. For example, if the associated Schubert system  $\Sigma(M, \mathcal{B})$  is totally solvable, then the question of finding a dense Schubert cell translates into a simple combinatorial problem; cf. [6] for more details.

Another criterium for identifying a dense Schubert cell is the following. Let  $U$  be a representation of dimension  $\underline{e}$ . We consider

$$C_U = \{U' \in \text{Gr}_{\underline{e}}(M) \mid U' \cong U\},$$

which is a smooth and irreducible subscheme of  $\text{Gr}_{\underline{e}}(M)$ , see [2, section 3.1] and [4, section 7]. In general, it can happen that  $\text{Gr}_{\underline{e}}(M)$  contains an infinite number of isomorphism types of  $\underline{e}$ -dimensional representations  $U$ . But in many interesting situations, e.g. for quivers of Dynkin type, certain representations of extended Dynkin type or if  $\text{Ext}(U, U) = 0$  for all  $U \in \text{Gr}_{\underline{e}}(M)$ , this number is finite.

Indeed, by [2, section 3.1] and [4, section 7], we see that  $\text{Ext}(U, U) = 0$  implies that every  $U \in \text{Gr}_{\underline{e}}(M)$  gives rise to an irreducible component  $\overline{C_U}$ . But up to isomorphism there are only finitely many exceptional representations of dimension  $\underline{e}$ . Using standard arguments from Auslander-Reiten theory, one also easily verifies that every indecomposable preprojective or regular representation of an extended Dynkin quiver admits only finitely many isomorphism classes of subrepresentations. Consequently, in all cases every irreducible component  $C_i$  of  $\text{Gr}_{\underline{e}}(M)$  is the closure of  $C_{U_i}$  for some  $U_i \in C_i$ . Note that, dually, every indecomposable preinjective representation admits only finitely many isomorphism classes of factors which makes it possible to suit the methods to this case.

Therefore, assuming the number of isomorphism classes in  $\text{Gr}_{\underline{e}}(M)$  is finite, the hypothesis of the theorem is satisfied for a Schubert cell  $C_I^M$  of  $\text{Gr}_{\underline{e}}(M)$  if  $C_I^M$  contains representations  $U_1, \dots, U_r$  that represent the irreducible components  $C_1, \dots, C_r$  of  $\text{Gr}_{\underline{e}}(M)$ , i.e.  $C_i = \overline{C_{U_i}}$  for  $i = 1, \dots, r$ . This is, in particular, the case if  $C_I^M$  contains representations of all isomorphism types that occur in  $\text{Gr}_{\underline{e}}(M)$ .

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